A SPINORIAL HAMILTONIAN APPROACH TO GRAVITY

JAMES D.E. GRANT

Abstract. We give a spinorial set of Hamiltonian variables for General Relativity in any dimension greater than 2. This approach involves a study of the algebraic properties of spinors in higher dimension, and of the elimination of second-class constraints from the Hamiltonian theory. In four dimensions, when restricted to the positive spin-bundle, these variables reduce to the standard Ashtekar variables. In higher dimensions, the theory can either be reduced to a spinorial version of the ADM formalism, or can be left in a more general form which seems useful for the investigation of some spinorial problems such as Riemannian manifolds with reduced holonomy group. In dimensions 0 (mod 4), the theory may be recast solely in terms of structures on the positive spin-bundle $\mathbb{V}^+$, but such a reduction does not seem possible in dimensions 2 (mod 4), due to algebraic properties of spinors in these dimensions.

1. Introduction

One of the central elements of the Ashtekar variables approach to canonical gravity [A1] is the projection, to a 3-dimensional hypersurface, of the natural connection on the positive spin-bundle $\mathbb{V}^+$ of a four manifold. This connection, originally introduced by Sen [S2], contains information about both the 3-dimensional spin-connection, and the extrinsic curvature of the hypersurface in a way which leads to a considerable simplification of the constraints of the Hamiltonian version of the theory. If we consider the Riemannian version of the theory, then Ashtekar's approach is very much based on the fact that the four-dimensional spin-group Spin_4 is not simple, but decomposes as $\mathrm{SU}(2) \times \mathrm{SU}(2)$. This decomposition means that the spin-connection decomposes into two independent SU(2) connections on the positive and negative chirality spin-bundles $\mathbb{V}^\pm$. (Similarly, there exists a reformulation of 3-dimensional gravity in SU(2) spinor form.) However, no such decomposition happens in higher even dimensions, with the connections on $\mathbb{V}^\pm$ in higher dimensions carrying all of the information of the full spin-connection. In odd dimensions, there is no chiral decomposition of the spin-bundle at all. It therefore seems that Ashtekar's approach is very much limited to 3 and 4-dimensional spaces.

Independently, however, Witten introduced a similar spinor connection in his proof of the Positive Energy Theorem [W1]. Although his argument is motivated by supergravity considerations, it is independent of the dimension of the spacetime (all that is required is that the hypersurfaces we consider admit a spin structure), and works equally well whether we work on the full spin-bundle or, in even dimensional spacetimes, restrict to the positive or negative spin-bundle.

The question we wish to analyse is whether one can develop a Hamiltonian theory based on Witten's connection in any dimension. Based on a (non-chiral) generalisation of the action introduced in four dimensions to describe Ashtekar's theory [JS, S1], we construct a Hamiltonian theory in any dimension (greater than 2) which reduces to Ashtekar's theory in dimension 4 when restricted to the positive spin-bundle. In general, setting up the theory requires an analysis of the algebraic structures on spin-bundles in general dimension, and the Hamiltonian theory contains

extra constraints and variables which do not naturally appear in the standard 4-dimensional theory. Many of these constraints and variables can be systematically removed from the theory, and reduce the theory to a minimal version which is independent of dimension. This theory contains both first and second-class constraints, but in dimension 3, or dimension 4 restricted to $\mathbb{V}^+$, all of the second-class constraints drop out of the theory, and the theory reduces to the relevant version of the Ashtekar theory. In all other cases, we can remove the second-class constraints by Dirac's procedure, leading to a theory with only first-class constraints. It is sometimes advantageous to work with the resulting formalism directly, but alternatively one can remove some of the first-class constraints and reduce the theory to a spinorial version of the ADM formalism.

The plan of the paper is as follows. We begin by explaining the spinorial action principle we will use. To begin with, this is simply a generalisation of the work of [JS, S1]. However, in higher dimensions, there are some subtleties with the equations of motion (in particular in dimensions 2 (mod 4)) which require a careful study of the properties of spin-bundles and Clifford algebras in different dimensions. We then proceed, in Section 3, with the Hamiltonian decomposition of the theory. Although, again, part of this work is standard, we find that the theory contains new constraints and variables in higher dimensions, with new sets of each appearing in dimensions 2 (mod 4) or 3 (mod 4) depending on the approach one adopts. In Section 3.2, we show how these extra constraints may be unravelled and removed from the theory, along with the extra variables. For completeness, it is then shown, in Section 4, how the resulting minimal theory may be reduced to the known Ashtekar version of the 3 and 4 dimensional theory. Finally, we consider the general version of this minimal theory in Section 5. We point out some circumstances in which it is useful to work with this theory directly, notably in the study of Riemannian metrics with reduced holonomy group. However, we also show how, if desired, the theory can be reduced to a spinorial analogue of the orthonormal-frame approach to the ADM formalism.

It should be noted that in higher even dimensions the natural connections on $\mathbb{V}^\pm$ are not independent, and carry all of the information of the space-time connection. It is therefore not to be expected that there will be any particular simplification in looking at a chiral version of the theory. In dimensions 0 (mod 4), it turns out that it is still possible to rewrite the theory simply in terms of the connection on $\mathbb{V}^+$, say, and it may be the case that such an approach would be useful if we were to couple the theory to chiral fermions. In dimensions 2 (mod 4), for algebraic reasons it does not seem possible to reduce the theory to $\mathbb{V}^+$. These algebraic arguments are essentially the same as those which suggest that coupling chiral fermions to gravity leads to formidable problems with the quantisation of the theory in these dimensions [AW].

2. Connections and Curvature

We work on a real spin manifold X of dimension $n \geq 3$. We assume that X carries a pseudo-Riemannian metric $\mathbf{g}$ of signature (r, s) and that, locally, we may introduce a pseudo-orthonormal basis $\{\boldsymbol{\epsilon}^A | A = 1 \dots n\}$ for the cotangent bundle T^*X in terms of which the metric may be written

$$\mathbf{g} = \eta_{AB} \boldsymbol{\epsilon}^A \otimes \boldsymbol{\epsilon}^B,$$

where the matrix η_{AB} takes the diagonal form

$$\eta_{AB} = \mathrm{diag}[\underbrace{1 \dots 1}_{r}, \underbrace{-1 \cdots -1}_{s}].$$

(Generally upper case letters $A, B, \ldots$ will denote internal $\mathrm{SO}(r,s)$ indices whilst lower case letters $a, b, \ldots$ will denote space-time coordinate indices. Similar conventions will be assumed for spatial indices, when we later consider Hamiltonian decompositions.) The spin connection Γ on the (pseudo)-orthonormal frame bundle is uniquely determined by torsion-free condition that the frame $\boldsymbol{\epsilon}$ is covariantly closed

$$d^{\boldsymbol{\Gamma}}\boldsymbol{\epsilon} = 0.$$

The fact that Γ is a connection on the (pseudo)-orthonormal frame bundle means that the connection automatically annihilates the metric

$$\nabla \mathbf{g} = 0.$$

We are free to make internal $\mathrm{SO}(r,s)$ transformations of the form

$$\boldsymbol{\epsilon} \mapsto \Lambda\boldsymbol{\epsilon},$$

where

$$\Lambda = \exp\left(\frac{1}{2}\alpha_{AB}M^{AB}\right),$$

and the generators M^{AB} form a representation of the Lie algebra of $\mathrm{SO}(r,s)$:

$$\left[M^{AB}, M^{CD}\right] = -\eta^{AC}M^{BD} + \eta^{AD}M^{BC} + \eta^{BC}M^{AD} - \eta^{BD}M^{AC}. \tag{2.1}$$

The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ is an algebra over $\mathbb{C}$, with identity Id, generated by T^*X, and may be viewed as the algebra generated by the skew-symmetrised products of objects γ^A which obey the relation

$$\gamma^A\gamma^B + \gamma^B\gamma^A = -2\eta^{AB}\mathrm{Id}. \tag{2.2}$$

The Clifford algebra over T^*X is canonically isomorphic as a vector space to the exterior algebra Λ^*X, so given any differential form $\boldsymbol{\lambda}$ on X, we may consider the corresponding section of the Clifford algebra bundle, denoted $\boldsymbol{\sigma}(\boldsymbol{\lambda})$. In particular, we may define $\gamma^A = \boldsymbol{\sigma}(\boldsymbol{\epsilon}^A)$. These objects may then be viewed as sections of the bundle $\mathrm{End}\mathbb{V}$ of endomorphisms of the spin-bundle $\mathbb{V}$.

The generators of the spin-1/2 representation of the $\mathfrak{so}(r,s)$ algebra (2.1) are

$$\Sigma^{AB} = -\frac{1}{4}\left[\gamma^A, \gamma^B\right].$$

The natural spinorial covariant derivative of a spinor field ψ is defined in terms of the image in the Clifford algebra of the spin connection $\boldsymbol{\Gamma}$ by

$$\begin{aligned} \nabla\psi &= d\psi + \frac{1}{2}\boldsymbol{\Gamma}_{AB}\Sigma^{AB}\psi \\ &= d\psi + \mathbf{A}\psi, \qquad \forall\psi \in \Gamma(\mathbb{V}), \end{aligned}$$

where

$$\mathbf{A} := \frac{1}{2}\Gamma_{AB}\Sigma^{AB}$$

is the spinor connection. The curvature of this connection, $\mathbf{F}$, is defined by the relation

$$\left([\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\psi = \mathbf{F}_{\mathbf{X}\mathbf{Y}}\psi, \qquad \forall\psi \in \Gamma(\mathbb{V}), \qquad \forall\mathbf{X}, \mathbf{Y} \in \Gamma(TX).$$

Defining the curvature of the spin connection

$$\mathbf{R} = d\boldsymbol{\Gamma} + \frac{1}{2}\left[\boldsymbol{\Gamma}, \boldsymbol{\Gamma}\right],$$

we can then identify the curvature $\mathbf{F}$ with its spinorial image

$$\mathbf{F}_{\mathbf{XY}} = \frac{1}{2}\mathbf{R}_{\mathbf{XY}AB}\Sigma^{AB}, \qquad \forall \mathbf{X}, \mathbf{Y} \in \Gamma(TX). \tag{2.3}$$

Standard γ-matrix techniques, and the cyclic identity for the Riemann tensor yield the identity

$$R_{abAB}\,\gamma^b\gamma^A\gamma^B = -2\,r_{ab}\gamma^b,$$

where r denotes the Ricci tensor of the metric g and we have defined the spacetime γ-matrices

$$\gamma^a := \gamma^A\epsilon_A{}^a.$$

This relation in turn implies that

$$R_{abAB}\,\gamma^a\gamma^b\gamma^A\gamma^B = -2\,s\,\mathrm{Id},$$

where $s = \mathrm{Tr}\ r$ is the scalar curvature of the metric $\mathbf{g}$. Therefore

$$\begin{aligned} s &= -\frac{1}{2D}\mathrm{Tr}\,\left(R_{abAB}\gamma^a\gamma^b\gamma^A\gamma^B\right) \\ &= \frac{2}{D}\mathrm{Tr}\,\left(F_{ab}\gamma^{ab}\right), \end{aligned}$$

where we have define the skew-symmetrised product of γ-matrices

$$\gamma^{a_1\ldots a_p} = \frac{1}{p!}\left[\gamma^{a_1}\ldots\gamma^{a_p} \pm \text{even and odd permutations}\right],$$

and $D = \dim\mathbb{V}$ is the rank of the spin bundle.

We can therefore rewrite the Einstein Hilbert action as

$$\begin{aligned} S_{EH} &= \frac{1}{16\pi G}\int_X g^{1/2}\,s\,d^nx \\ &= \frac{1}{8\pi G\,D}\int_X g^{1/2}\,\mathrm{Tr}\,\left[F_{ab}\gamma^{ab}\right]\,d^nx. \end{aligned}$$

It will be useful to define units in which

$$4\pi G\,D = 1$$

in which case we have

$$S_{EH} = \frac{1}{2}\int_X g^{1/2}\,\mathrm{Tr}\,\left[F_{ab}\gamma^{ab}\right]\,d^nx. \tag{2.4}$$

This action has been considered in the special case of dimension 4 with the connection restricted to the positive chirality spin bundle in connection with Ashtekar variables [JS, S1].

2.1. **Equations of Motion.** Consider now the equations of motion that follow from the action (2.4). We take the connection $\mathbf{A}$ and the spacetime γ-matrices γ^a as the independent variables with the inverse spacetime metric being constructed from the latter by the relation

$$g^{ab} = -\frac{1}{D}\mathrm{Tr}\,(\gamma^a\gamma^b).$$

The variation of the connection $\mathbf{A}$ tells us that

$$D_b\left(g^{1/2}\gamma^{ab}\right) = 0. \tag{2.5}$$

This equation, by itself, is not enough to uniquely determine the connection. This, however, is not a problem unique to our spinorial approach. If one considers the standard Palatini approach to the Einstein-Hilbert action, then the equations which follow from variation of the connection is supposed to uniquely determine the connection as the Levi-Civita connection. On closer inspection,

however, this turns out not to be the case. If one, a priori, assumes the connection is torsion-free, then the equations of motion tell us that the connection is metric, and vice versa. However, if we start with a completely general connection, the equations of motion in the Palatini formalism are insufficient to uniquely determine the connection.

It is therefore important to consider what we would like to demand of a connection, and what further conditions we must impose, by hand, on the connection so that the equations of motion uniquely confine it to be the spin-connection.

The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ has a unique irreducible representation on $\mathbb{C}^D$ when n is even, and two inequivalent irreducible representations on $\mathbb{C}^D$ when n is odd, where

$$D = \begin{cases} 2^{n/2} & n \text{ even} \\ 2^{(n-1)/2} & n \text{ odd.} \end{cases}$$

Therefore, assume that have an irreducible representation of our Clifford algebra on $\mathbb{V} \cong \mathbb{C}^D$. The elements of the Clifford algebra are then represented as endomorphisms of $\mathbb{V}$ so, relative to any basis for $\mathbb{V}$, would correspond to elements of $\mathbb{C}(D)$, the set of $D \times D$ complex matrices.

Given an irreducible representation of the algebra on a space $\mathbb{V}$, we may construct bi-linear forms

$$\begin{aligned} {}^{\pm}\epsilon : \mathbb{V} \otimes \mathbb{V} \to \mathbb{C}, \qquad & {}^{\pm}\epsilon^* : \mathbb{V}^* \otimes \mathbb{V}^* \to \mathbb{C} \qquad & n \text{ even,} \\ \epsilon : \mathbb{V} \otimes \mathbb{V} \to \mathbb{C}, \qquad & \epsilon^* : \mathbb{V}^* \otimes \mathbb{V}^* \to \mathbb{C} \qquad & n \text{ odd,} \end{aligned}$$

with the symmetry properties shown in Table 1 [PR]. (In odd dimensions, ϵ shall denote the one of ${}^{\pm}\epsilon$ which is non-vanishing.)

$n \pmod 8$	${}^{+}\epsilon$	${}^{-}\epsilon$
0	Symmetric	Symmetric
1	zero	Symmetric
2	Skew-symmetric	Symmetric
3	Skew-symmetric	zero
4	Skew-symmetric	Skew-symmetric
5	zero	Skew-symmetric
6	Symmetric	Skew-symmetric
7	Symmetric	zero

Table 1. Symmetries of ${}^{\pm}\epsilon$ in various dimensions

One can show that the forms ${}^{\pm}\epsilon$ have the properties that, for $p = 1, \dots n$

$${}^{\pm}\epsilon(\lambda, \gamma^{A_1 \dots A_p} \phi) = (\mp 1)^p \, (-1)^{\frac{p(p-1)}{2}} \, {}^{\pm}\epsilon(\gamma^{A_1 \dots A_p} \lambda, \phi), \qquad \forall \lambda, \phi \in \mathbb{V}. \tag{2.6}$$

If we now consider a spin manifold X, with metric $\mathbf{g}$, then all of the above algebraic considerations carry across to the Clifford algebra bundle over X. This is the bundle generated by Clifford multiplication from the cotangent bundle T^*X, and the representation space $\mathbb{V}$ becomes the spin bundle, the sections of which are spinor fields. Since the maps ${}^{\pm}\epsilon$ are suitably equivariant under $\mathrm{Spin}_{r,s}$ transformations, they carry across directly to corresponding forms on the spin-bundles.

Given an orthonormal frame for T^*X, there is a natural connection on T^*X, the spin-connection. One can lift this connection to a unique connection on the spin-bundle $\mathbb{V}$. We wish to consider a minimal set of spinorial conditions we can impose on a connection on $\mathbb{V}$ which will uniquely define it to be this image of the spin-connection. Given a connection on T^*X, we can extend this to a connection on Λ^*X. One would then like to define a connection on the Clifford algebra bundle which commutes with this map $\boldsymbol{\sigma}$ introduced above

$$(\nabla_{\mathbf{X}} \circ \boldsymbol{\sigma})\,\boldsymbol{\lambda} = \boldsymbol{\sigma}\,(\nabla_{\mathbf{X}}\boldsymbol{\lambda})\,. \tag{2.7}$$

In colloquial terms, this means that if we view the collection of γ-matrices as a section of $TX \otimes \mathrm{End}(\mathbb{V}) \cong TX \otimes \mathbb{V}^* \otimes \mathbb{V}$ then, given the connection on TX, we wish to arrange the connection on $\mathbb{V}$ so that this section is covariantly constant. This does not uniquely determine the connection. However, if we impose the additional requirement on the connection on $\mathbb{V}$ that it annihilates the forms ${}^{\pm}\epsilon$, this uniquely determines the connection on $\mathbb{V}$ to be the image of the spin-connection defined above.

What we need to know, however, is the minimal set of conditions we must impose on a connection on $\mathbb{V}$ in order that when combined with the equation of motion (2.5) the connection is uniquely determined to be the image of the spin connection. One requirement would be that the connection should annihilate the ${}^{\pm}\epsilon$

$$\begin{aligned} \nabla^{\pm}\epsilon &= 0 && n \text{ even},\\ \nabla\epsilon &= 0 && n \text{ odd}. \end{aligned} \tag{2.8}$$

The question is to what extent this condition determines the connection. If we consider two connections ∇ and ∇' on $\mathbb{V}$, then

$$\nabla'_{\mathbf{X}}\psi - \nabla_{\mathbf{X}}\psi = <\mathbf{T}, \mathbf{X}>\psi, \qquad \forall \mathbf{X} \in \Gamma(TX), \qquad \forall \psi \in \Gamma(\mathbb{V}),$$

where $\mathbf{T}$ is a section of $\Lambda^1(X) \otimes \mathrm{End}(\mathbb{V})$ that transforms under the adjoint representation under the $\mathrm{Spin}_{r,s}$ action on $\mathbb{V}$. If we assume that both connections annihilate the forms ${}^{\pm}\epsilon$, then we find that we require

$${}^{\pm}\epsilon(\lambda, <\mathbf{T}, \mathbf{X}>\phi) + {}^{\pm}\epsilon(<\mathbf{T}, \mathbf{X}>\lambda, \phi) = 0, \qquad \forall \lambda, \phi \in \Gamma(\mathbb{V}), \qquad \forall \mathbf{X} \in \Gamma(TX).$$

From Equation (2.6), we therefore deduce that $<\mathbf{T}, \mathbf{X}>$ is an section of $\mathrm{Im}_{\sigma}\left(\Lambda^2 \oplus \Lambda^6 \oplus \dots\right) \subset \mathbb{C}\mathrm{l}_{r,s}$ for all vector fields $\mathbf{X}$. Therefore

$$\mathbf{T} \in \Gamma\left(\Lambda^1(X) \otimes \mathrm{Im}_{\sigma}\left(\Lambda^2 \oplus \Lambda^6 \oplus \dots\right)\right). \tag{2.9}$$

(The expansion on the right-hand-side of this equation terminates when we reach the highest integer $4k+2$ less than or equal to n.) We should perhaps note that the forms ${}^{\pm}\epsilon$ are not uniquely determined by the Clifford algebra, but only determined up to a scale. Therefore a choice of this scale is implicit in Equation (2.8). One could equivalently work without fixing this scale, imposing only the existence of 1-forms ${}^{\pm}\boldsymbol{\lambda}$ with the property that $\nabla^{\pm}\epsilon = {}^{\pm}\boldsymbol{\lambda} \otimes {}^{\pm}\epsilon$ on the connection. If we then impose the additional condition that connection is trace-free (i.e. is an $\mathrm{SL}(\mathbb{V})$ connection), then we recover the result (2.9).

We now wish to impose the further condition on our connection that it satisfies Equation (2.5), which followed from our action principle considerations. If we assume that the connections ∇ and

∇' obey this equation, then the field $\mathbf{T}$ obeys the condition

$$\left[T_a, \gamma^{ab}\right] = 0. \tag{2.10}$$

Since pure elements of order 2 in the Clifford algebra generate $\mathrm{Spin}_{r,s}$ transformations, Equation (2.10) must be separately satisfied by each of the components in the Clifford algebraic expansion of $\mathbf{T}$ given in Equation (2.9). First consider dimensions $n \neq 2 \pmod 4$. Denote the part of $\mathbf{T}$ of order $m = 4k+2$ in the Clifford algebra by $\mathbf{T}_m$. This can be viewed as a section of $\wedge^1(X) \otimes \wedge^m(X) \cong \wedge^{m+1}(X) \oplus \wedge^{m-1}(X) \oplus S_0^{m+1}(X)$, where the separate terms correspond to the totally skew-symmetric, trace, and trace-free parts of $\mathbf{T}_m$, respectively. We then use the simple result:

Lemma 1.

$$\left[\gamma_{AB}, \gamma^{C_1 C_2 \ldots C_p}\right] = 4\, p\, \delta^{[C_1}_{[A} \gamma_{B]}{}^{C_2 \ldots C_p]}, \qquad p = 1, \ldots n.$$

The proof of this result is a straightforward application of definitions, and so will be omitted. With this result, it follows from the relation (2.10) that the $\wedge^{m+1}(X)$ and $\wedge^{m-1}(X)$ parts of $\mathbf{T}_m$ vanish. The remaining equations are then inconsistent with the symmetries required of an element of S_0^m (this is a generalisation of the argument that any $(0,3)$ tensor, $\mathbf{a}$, with the symmetry property $\mathbf{a}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{a}(\mathbf{y},\mathbf{x},\mathbf{z}) = -\mathbf{a}(\mathbf{x},\mathbf{z},\mathbf{y})$ must vanish identically), and so the only solution of our requirements is $\mathbf{T}_m = 0$, and therefore $\mathbf{T} = 0$. Therefore, if $n \neq 2 \pmod 4$, the connection which annihilates the forms ${}^{\pm}\epsilon$ and which satisfies the equations of motion (2.5) is unique, and must be the pull-back to the spin-bundle of the spin-connection, as required.

In dimension $n = 4k+2$, however, this argument breaks down. If we consider the final term in the expansion (2.9), we have the possibility

$$\mathbf{T} = \boldsymbol{\phi} \otimes \omega$$

for any 1-form field $\boldsymbol{\phi}$, where

$$\omega = i^{\alpha} \gamma^1 \cdots \gamma^n, \qquad \alpha = \left[\frac{n+1}{2}\right] \tag{2.11}$$

is the volume element on the Clifford algebra [LM]. With respect to a general connection of this form, the γ-matrices are not covariantly constant, although a skew-symmetrised product of an even number of γ-matrices is covariantly constant. Since such a term has no analogue in a tensorial approach, it does not seem to have any straightforward geometrical interpretation.

If we wish to reproduce Einstein-Hilbert gravity it is therefore necessary, in these dimensions, to impose an additional condition on the spinor connection to remove this extra freedom. We know, from above, that any connection which preserves the forms ${}^{\pm}\epsilon$ must be in the image of $\Lambda^2 \oplus \Lambda^6 \oplus \cdots \oplus \Lambda^n$. The most direct way to remove the extra freedom of equation (2.12) is therefore to impose by hand the condition that there is no Λ^n term:

$$\mathbf{A} \in \Gamma\left(\Lambda^1(X) \otimes \mathrm{Im}_{\sigma}\left(\Lambda^0(X) \oplus \Lambda^1(X) \oplus \cdots \oplus \Lambda^{n-1}(X)\right)\right) \qquad n \equiv 2 \pmod 4. \tag{2.12}$$

This is equivalent to imposing the condition

$$\mathrm{Tr}\,(\mathbf{A}\omega) = 0$$

on the connection. Only once this extra degree of freedom has been removed from the theory do we find that the connection defined by Equations (2.9) and (2.10) is the connection we require.

Note that, in principle, the algebraic conditions (2.8) and (2.12) on the connection could be imposed as extra primary constraints in the Lagrangian approach we have taken. Since these constraints have no dynamics, however, the resulting theory will be identical with the one which results by simply assuming that the connection obeys these constraints identically. For simplicity, we shall adopt the latter approach.

In even dimensions, we can use ω to split up $\mathbb{V}$ into chiral parts $\mathbb{V}^{\pm}$. A significant difference between dimensions $0 \pmod 4$ or $2 \pmod 4$ is that we can arrange that

$$\begin{aligned} &{}^{+}\epsilon : \mathbb{V}^{+} \otimes \mathbb{V}^{+} \to \mathbb{C}, && {}^{-}\epsilon : \mathbb{V}^{-} \otimes \mathbb{V}^{-} \to \mathbb{C}, && n \equiv 0 \pmod 4, \\ &{}^{+}\epsilon : \mathbb{V}^{+} \otimes \mathbb{V}^{-} \to \mathbb{C}, && {}^{-}\epsilon : \mathbb{V}^{-} \otimes \mathbb{V}^{+} \to \mathbb{C}, && n \equiv 2 \pmod 4. \end{aligned}$$

Therefore ${}^{\pm}\epsilon$ define isomorphisms

$$\begin{aligned} &\mathbb{V}^{\pm} \cong (\mathbb{V}^{\pm})^*, && n \equiv 0 \pmod 4, \\ &\mathbb{V}^{\pm} \cong (\mathbb{V}^{\mp})^*, && n \equiv 2 \pmod 4. \end{aligned}$$

The fact that these isomorphisms are between different spin-bundles in dimensions $2 \pmod 4$ means that in these dimensions we cannot reduce the problem to a single spin-bundle, and a chiral spinorial formalism does not seem to be possible. In dimensions $0 \pmod 4$, however, it is possible to set up the theory completely on, for example, the positive spin-bundle $\mathbb{V}^{+}$. In odd dimensions, the volume element is a central element of the Clifford algebra, so if we consider irreducible spin-bundles, there is no similar reduction.

Returning to the equations of motion, if we assume that we have imposed these extra conditions in such a way that the connection $\mathbf{A}$ is identified with the spin connection, and therefore its curvature is identified with the curvature of the spin-connection as in Equation (2.3), then varying the γ-matrices in the action (2.4), we find,

$$\delta S_{EH} = \frac{1}{2} \int_X g^{1/2} \mathrm{Tr} \left((\delta\gamma^a) \gamma_b ({r_a}^b - \frac{1}{2} s {\delta_a}^b) \right) d^n x.$$

Therefore the equation of motion tells us that the metric $\mathbf{g}$ satisfies the vacuum Einstein equations

$$\mathbf{r} = \frac{s}{2} \mathbf{g}.$$

As such, as long as we impose the condition that the connection $\mathbf{A}$ annihilates the natural bilinear forms on the spin bundle, along with the extra condition of no volume terms in dimensions $2 \pmod 4$, then the equations of motion which follow from our version of the Einstein-Hilbert action are equivalent to the vacuum Einstein equations.

3. Riemannian Hamiltonian Decomposition

We now wish to consider the Hamiltonian version of the above theory. For simplicity, we will consider metrics of Riemannian signature, although other signatures can be treated similarly. Our treatment initially follows the standard approach in four dimensions [JS, S1].

We consider a suitable open set $U \subset X$ of our manifold which we assume to be foliated by a 1-parameter family of leaves Σ of dimension $(n-1)$. Introducing a parameter t to label the different leaves of this foliation, and local coordinates $\{x^i | i = 1, \ldots, n-1\}$ on Σ, we may decompose the metric in standard Hamiltonian form

$$g = \boldsymbol{\epsilon}^0 \otimes \boldsymbol{\epsilon}^0 + \delta_{IJ} \boldsymbol{\epsilon}^I \otimes \boldsymbol{\epsilon}^J,$$

where we take

$$\begin{aligned}\boldsymbol{\epsilon}^0 &= N dt,\\ \boldsymbol{\epsilon}^I &= e_i{}^I\left(dx^i + N^i dt\right).\end{aligned}$$

(Upper case indices $I, J, \ldots$ take values $1, 2, \ldots, n-1$.) The dual basis takes the form

$$\begin{aligned}\mathbf{e}_0 &= N^{-1}\left(\partial_t - N^i\partial_i\right),\\ \mathbf{e}_I &= e_I{}^i\partial_i.\end{aligned}$$

The induced metric (first fundamental form) on Σ will be denoted $\mathbf{q}$, and has components

$$q_{ij} = \delta_{IJ} e_i{}^I e_j{}^J$$

with respect to the coordinates introduced above.

At this point it is useful to recall the Clifford algebra isomorphism

$$\mathbb{C}\mathrm{l}_n^{\mathrm{even}} \cong \mathbb{C}\mathrm{l}_{n-1}.$$

In our context, this means that we may define the $(n-1)$-dimensional γ-matrices by

$$\Gamma^I := \gamma^{0I}, \qquad I = 1, \ldots, n-1,$$

which generate the algebra $\mathbb{C}\mathrm{l}_{n-1}$. We also define the spatial γ-matrices

$$\Gamma^i := \Gamma^I\, e^i_I,$$

and their skew-symmetrised products $\Gamma^{ij\ldots k}$.

If we now insert the decomposition of the metric into the Einstein-Hilbert action, it takes the form

$$S = \int_X q^{1/2}\mathrm{Tr}\left[\Gamma^I < \mathcal{L}_t\boldsymbol{\gamma}, \mathbf{e}_I > -\Gamma^i D_i^{\gamma} A_t + \Gamma^i F^{\gamma}_{ij} N^j + \frac{N}{2} F^{\gamma}_{ij}\Gamma^{ij}\right] dt\, d^{n-1}x.$$

In this equation we have defined the connection

$$\boldsymbol{\gamma} = p_\Sigma(\mathbf{A})$$

as the pull-back of the connection $\mathbf{A}$ to the surface Σ. The curvature of this connection, $\boldsymbol{\gamma}$ is denoted $\mathbf{F}^\gamma$, and we have defined the covariant derivative of the field A_t by

$$D_i A_t = \partial_i A_t + \left[\gamma_i, A_t\right].$$

We also, for simplification later, introduce a densitised version of the function N by defining

$$\underline{N} := q^{-1/2} N,$$

where

$$q := \left|\det\left(q_{ij}\right)\right|.$$

In order to proceed with the Hamiltonian decomposition, we introduce momenta conjugate to all of the dynamical variables. from the form of the Lagrangian above, we deduce that the

momenta conjugate to the variables $(\gamma_i, {e_I}^i, A_t, \underline{N}, N^i)$, in order, take the form:

$$\begin{aligned} \pi^i &= \tilde{\sigma}^i, \\ {\pi^I}_i &= 0, \\ \pi^t &= 0, \\ \underline{\pi} &= 0, \\ \pi_i &= 0, \end{aligned}$$

where we have defined the densitised γ-matrices

$$\tilde{\sigma}^i := q^{1/2}\Gamma^i.$$

In Dirac's terminology [D], we therefore have the primary constraints of the theory

$$\phi^i = \pi^i - \tilde{\sigma}^i, \tag{3.1a}$$
$${\phi^I}_i = {\pi^I}_i, \tag{3.1b}$$
$$\phi^t = \pi^t, \tag{3.1c}$$
$$\underline{\phi} = \underline{\pi}, \tag{3.1d}$$
$$\phi_i = \pi_i. \tag{3.1e}$$

The total Hamiltonian H_T of the theory is now the sum of the canonical Hamiltonian $H_c \sim p\dot{q} - L$ and primary constraints with suitable Lagrange multipliers

$$H_T = -\int_\Sigma \mathrm{Tr}\left[A_t D_i^\gamma \tilde{\sigma}^i + F_{ij}^\gamma \tilde{\sigma}^i N^j + \frac{1}{2}\underline{N} F_{ij}^\gamma \tilde{\sigma}^i \tilde{\sigma}^j\right] \\ + \int_\Sigma \mathrm{Tr}\left(\lambda_i \phi^i\right) + {\lambda_I}^i {\phi^I}_i + \mathrm{Tr}\left(\lambda_t \phi^t\right) + \underline{\lambda}\underline{\phi} + \lambda^i \phi_i.$$

Time evolution is generated by Poisson Brackets with this Hamiltonian, with $\mathcal{L}_t f = [f, H_T]$ with f any function on the phase space, and where the variables $q = (\gamma_i, {e_I}^i, A_t, \underline{N}, N^i)$ and conjugate momenta $p = (\pi^i, {\pi^I}_i, \pi^t, \underline{\pi}, \pi_i)$ obey the heuristic Poisson Bracket relations

$$[q, p] = 1. \tag{3.2}$$

3.1. **Secondary Constraints.** The primary constraints of the theory (3.1) must be preserved under time evolution. In order that the constraints $\phi^t, \phi_i, \underline{\phi}$ be preserved by evolution, their Poisson Bracket with H_T must be a sum of constraints. This leads to the secondary constraints of the theory

$$\chi_1 := D_i^\gamma \tilde{\sigma}^i \approx 0, \tag{3.3a}$$
$$\chi_{2i} := -\mathrm{Tr}\left(F_{ij}^\gamma \tilde{\sigma}^j\right) \approx 0, \tag{3.3b}$$
$$\chi_3 := \frac{1}{2}\mathrm{Tr}\left(F_{ij}^\gamma \tilde{\sigma}^i \tilde{\sigma}^j\right) \approx 0. \tag{3.3c}$$

The preservation of the constraints $\chi_1, \chi_{2i}, \chi_3$ do not lead to any new secondary constraints of the theory, but simply place restrictions on the Lagrange multipliers

$$\begin{aligned}
\mathcal{L}_t\chi_1 &\approx \left[\lambda_i, \tilde{\sigma}^i\right] \approx 0,\\
\mathcal{L}_t\chi_{2i} &\approx -\mathrm{Tr}\,\left[\left(D_i^\gamma\lambda_j - D_j^\gamma\lambda_i\right)\tilde{\sigma}^j\right] \approx 0,\\
\mathcal{L}_t\chi_3 &\approx \frac{1}{2}\mathrm{Tr}\,\left[D_i^\gamma\lambda_j\left[\tilde{\sigma}^i, \tilde{\sigma}^j\right]\right] \approx 0.
\end{aligned}$$

If we consider the equations of motion for the corresponding variables $(A_t, N^i, \underline{N})$, we find

$$\mathcal{L}_tA_t = \lambda_t, \qquad \mathcal{L}_t\underline{N} = \underline{\lambda}, \qquad \mathcal{L}_tN^i = \lambda^i.$$

Since the constraints $\phi^t, \phi_i, \underline{\phi}$ have vanishing Poisson Brackets with all the other constraints, we may drop the momenta $\pi^t, \pi_i, \underline{\pi}$, and the multipliers $\lambda_t, \lambda^i, \underline{\lambda}$, and simply view $A_t, N^i, \underline{N}$ as Lagrange multipliers enforcing the constraints (3.3).

Eliminating these redundant variables and constraints implies that we are left with the dynamical variables of the theory

$$(\gamma_i, \pi^i; {e_I}^i, {\pi^I}_i),$$

and the constraints

$$\phi^i = \pi^i - \tilde{\sigma}^i, \tag{3.4a}$$

$${\phi^I}_i = {\pi^I}_i, \tag{3.4b}$$

$$\chi_1 = D_i^\gamma\tilde{\sigma}^i, \tag{3.4c}$$

$$\chi_{2i} = -\mathrm{Tr}\,\left(F_{ij}^\gamma\tilde{\sigma}^j\right), \tag{3.4d}$$

$$\chi_3 = \frac{1}{2}\mathrm{Tr}\,\left(F_{ij}^\gamma\tilde{\sigma}^i\tilde{\sigma}^j\right). \tag{3.4e}$$

The Hamiltonian of the theory is simply the sum of these constraints multiplied by the appropriate Lagrange multipliers

$$H_T = -\int_\Sigma\left(\mathrm{Tr}\,[A_t\chi_1] + N^i\chi_{2i} + \underline{N}\chi_3\right) + \int_\Sigma\left(\mathrm{Tr}\,\left(\lambda_i\phi^i\right) + {\lambda_I}^i{\phi^I}_i\right).$$

Since the Hamiltonian is a sum of constraints, the preservation of these constraints under time evolution reduces to a problem concerning the Poisson Brackets of the constraints. If a constraint is first-class, then it will automatically be preserved by the evolution, whereas if it is second-class, its preservation will place restrictions on the Lagrange multipliers. In neither case will preservation under time evolution introduce new constraints, so the only constraints of the theory are those given in (3.4).

3.2. **Removal of Higher Order Constraints.** We know from the arguments of Section 2.1 that the connection $\mathbf{A}$ must obey the extra geometrical constraint that it annihilates the forms ${}^\pm\epsilon$ on the spin-bundle. In terms of the Hamiltonian theory, this implies that the spinorial quantities (A_i, π^i, A_t) take values in $\mathrm{Im}_\sigma(\Lambda^2(X) \oplus \Lambda^6(X) \oplus \dots) \subset \mathbb{C}\mathrm{l}_n$, where the expansion terminates at $\Lambda^{4k+2}(X)$ for the largest value of k with $4k+2 \leq n$. This, in turn, implies that the constraints (ϕ^i, χ_1) take values in the same space. As such, we see that new constraints enter the theory in each dimension 2 (mod 4). In line with the arguments of Section 2.1, however, we know that in these dimensions we must impose the extra condition that the Clifford algebra expansion of the spinor-connection (and therefore the conjugate momentum) must not contain any multiple of the

volume form in order to recover standard Einstein-Hilbert gravity. Therefore, we must also, by hand, remove the corresponding constraints which arise. As such, the new constraints really only come into effect in dimensions 3 (mod 4).

It now becomes useful to divide the constraints (ϕ^i, χ_1) and the variables (A_i, π^i, A_t) into a pure second order part taking values in $\mathrm{Im}_\sigma \Lambda^2(X) \subset \mathbb{C}\mathrm{l}_n$ and a part taking values in the higher order space $\mathrm{Im}_\sigma(\Lambda^6(X) \oplus \Lambda^{10}(X) \oplus \dots) \subset \mathbb{C}\mathrm{l}_n$. Further decomposing using the isomorphism $\mathbb{C}\mathrm{l}_n^{\mathrm{even}} \cong \mathbb{C}\mathrm{l}_{n-1}$ introduced in Section 3, we define

$$\begin{aligned} \gamma_i &:= A_i + \tilde{A}_i, \\ \pi^i &:= \Pi^i + \tilde{\Pi}^i, \\ \phi^i &:= \Phi^i + \tilde{\Phi}^i, \\ \chi_1 &:= \Psi + \tilde{\Psi}, \end{aligned}$$

where

$$A_i, \Pi^i, \Phi^i, \Psi \in \mathrm{Im}_\sigma(\Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma)), \qquad \tilde{A}_i, \tilde{\Pi}^i, \tilde{\Phi}^i, \tilde{\Psi} \in \mathrm{Im}_\sigma(\Lambda^5(\Sigma) \oplus \Lambda^6(\Sigma) \oplus \dots).$$

We can therefore write

$$\begin{aligned} \Phi^i &= \Pi^i - \tilde{\sigma}^i, & \tilde{\Phi}^i &= \tilde{\Pi}^i, \\ \Psi &= \partial_i \tilde{\sigma}^i + \left[A_i, \tilde{\sigma}^i\right], & \tilde{\Psi} &= \left[\tilde{A}_i, \tilde{\sigma}^i\right]. \end{aligned}$$

Generally, we will refer to the quantities of higher order in the Clifford algebra i.e. $\tilde{A}_i, \tilde{\Pi}^i, \tilde{\Phi}^i, \tilde{\Psi}$ as "higher order" quantities. Our first goal is to show that the higher order variables and constraints may be removed from the theory, leaving a theory with only the first and second order quantities in the Clifford algebra, i.e. A_i, Π^i, Φ^i, Ψ. To this end, we make use of the following straightforward result:

Lemma 2. *The trace of the skew-symmetrised γ-matrices*

$$\mathrm{Tr}\,\left(\gamma^{A_1 \dots A_{2l}} \gamma_{B_1 \dots B_{2m}}\right)$$

vanishes unless $l = m$ and the indices $A_1, \dots A_{2l}$ are a permutation of the indices $B_1 \dots B_{2m}$. In particular,

$$\mathrm{Tr}\,\left(\gamma^{A_1 \dots A_k} \gamma_{B_1 \dots B_k}\right) = (-1)^{\frac{k(k+1)}{2}}\, D\, k!\, \delta^{[A_1}_{B_1} \cdots \delta^{A_k]}_{B_k},$$

where D is the rank of the spin-bundle.

Assuming linear independence of the vector fields $\mathbf{e}_I$ on Σ, then $\mathbb{C}\mathrm{l}_{n-1}$ will be spanned by the elements $\tilde{\sigma}^i$ and their skew-symmetrised products. We may therefore decompose each of the higher order spinorial quantities above into pure constituent parts in the Clifford algebra using the $\tilde{\sigma}^i$. We define

$$\begin{aligned} \phi^{ij_1 \cdots j_m} &:= \mathrm{Tr}\,\left[\tilde{\Phi}^i \tilde{\sigma}^{[j_1} \dots \tilde{\sigma}^{j_m]}\right], \\ \psi^{j_1 \cdots j_m} &:= \mathrm{Tr}\,\left[\tilde{\Psi} \tilde{\sigma}^{[j_1} \dots \tilde{\sigma}^{j_m]}\right]. \end{aligned}$$

where $m = 4k + 1$ or $m = 4k + 2$ with $k = 1, 2, \dots$.

These constraints are not all first-class, and our first task is to identify the second-class constraints and remove them by Dirac's procedure. In order to do this, we consider the constraints

$\phi^{ij_1\cdots j_m}$ in terms of SO_{n-1} representation theory. We can represent these constraints as an element of $\wedge^1(\Sigma)\otimes\wedge^m(\Sigma)$ and use the SO_{n-1} decomposition

$$\wedge^1(\Sigma)\otimes\wedge^m(\Sigma)\cong\wedge^{m+1}(\Sigma)\oplus S_0^{m+1}(\Sigma)\oplus\wedge^{m-1}(\Sigma). \tag{3.5}$$

In this decomposition, $\phi\in\wedge^1(\Sigma)\otimes\wedge^m(\Sigma)$ decomposes into a totally skew-symmetric part, denoted $\wedge^{m+1}[\phi]$, and a part of mixed symmetry, $S^{m+1}[\phi]$. This second part may then be reduced to irreducible components consisting of a trace-free element $S_0^{m+1}[\phi]\in S_0^{m+1}(\Sigma)$, and a trace part, which may then be identified with an element of $\wedge^{m-1}(\Sigma)$, denoted $\wedge^{m-1}[\mathrm{Tr}\,\phi]$.

Similarly, $\tilde{\Psi}\in\mathrm{Im}_\sigma(\Lambda^5(\Sigma)\oplus\Lambda^6(\Sigma)\oplus\dots)$ and we denote the separate elements of this decomposition by $\wedge^m[\psi], m=4k+1,4k+2$.

Recall that Dirac's procedure for eliminating second-class constraints from finite dimensional systems consists of finding a set of second class constraints $\{\alpha_i|i=1,\dots,k\}$ with the property that the matrix of Poisson Brackets $C_{ij}:=[\alpha_i,\alpha_j]$ is of maximal rank [D]. One then defines the Dirac bracket of two arbitrary functions f and g on the phase space by the modified relation

$$[f,g]^*:=[f,g]-[f,\alpha_i]\left(C^{-1}\right)^{ij}[\alpha_j,g]\,. \tag{3.6}$$

The Dirac bracket automatically has the property that $[f,\alpha_i]^*=0$ for any function f on the phase space. Similarly, given a function f on the phase space, we define a new function,

$$f^*:=f-[f,\alpha_i]\left(C^{-1}\right)^{ij}\alpha_j. \tag{3.7}$$

The constraints automatically vanish, in the sense that $\alpha_i^*=0$. The assumption of the Dirac approach is that the dynamics of the theory, with second class constraints removed, is generated by a Hamiltonian H^* via Dirac Brackets

$$\frac{d}{dt}f^*:=[f^*,H^*]^*\,.$$

In practice, this means we remove the redundant variables of the theory by imposing the constraints as identities, and then define the modified Poisson Brackets of the remaining quantities to be the Dirac Bracket.

The procedure generalises to field theories with constraints in the obvious fashion (although a choice of boundary conditions is generally involved in constructing the inverse matrix C^{-1}). In our case, the goal is therefore to find pairs of families of second-class constraints which are canonically conjugate, in the sense that the matrix of Poisson Brackets of the constraints is of maximal rank. Using the Poisson Brackets (3.2), we find that two such families of conjugate second-class constraints are given by

$$\wedge^{4k+1}[\psi]\leftrightarrow\wedge^{4k+1}[\mathrm{Tr}\,\phi], \tag{3.8}$$

$$\wedge^{4k+2}[\psi]\leftrightarrow\wedge^{4k+2}[\phi]. \tag{3.9}$$

These constraints have vanishing Poisson Brackets with all other variables and constraints of the theory, and they simply constrain the variables $(\wedge^{4k+1}[\tilde{A}],\wedge^{4k+2}[\tilde{A}])$ and their conjugate momenta $(\wedge^{4k+1}[\mathrm{Tr}\,\tilde{\Pi}],\wedge^{4k+2}[\tilde{\Pi}])$ to vanish. Therefore, when we construct the Dirac brackets as in Eq. (3.6), these constraints have vanishing Dirac bracket with all other functions on the phase space. All other Dirac Brackets remain equal to the original Poisson Brackets. Similarly, when we redefine functions on the phase space as in Eq. (3.7), this sets the variables $(\wedge^{4k+1}[\tilde{A}],\wedge^{4k+2}[\tilde{A}])$ and their conjugate momenta $(\wedge^{4k+1}[\mathrm{Tr}\,\tilde{\Pi}],\wedge^{4k+2}[\tilde{\Pi}])$ identically to zero, whilst all other functions

on the phase space remain unchanged. Therefore the constraints (3.8) and (3.9) simply set the variables $(\wedge^{4k+1}[\tilde{A}], \wedge^{4k+2}[\tilde{A}])$ and momenta $(\wedge^{4k+1}[\mathrm{Tr}\,\tilde{\Pi}], \wedge^{4k+2}[\tilde{\Pi}])$ identically to zero, so that these variables and momenta are removed completely from the theory.

Carrying out this procedure for each value of $k \geq 1$ completely removes all of the higher order constraints $\tilde{\Psi}$ from the theory, leaving us with only the first and second order part of χ_1, denoted Ψ above. The remaining higher order parts of the constraint ϕ^i impose that the remaining parts of the momenta $\tilde{\Pi}$ vanish. Preservation of these constraints under time evolution then places restrictions on the Lagrange multipliers of the theory. These restrictions, modulo the second-class constraints discussed above, when combined with the explicit form of the metric imply that

$$\left[\tilde{A}_a, \gamma^{ab}\right] = 0.$$

It is straightforward to show, in dimensions $n \neq 2 \pmod 4$, that this implies $\tilde{A} = 0$. In dimensions $n = 2 \pmod 4$ these conditions imply the $\tilde{A}$ is a multiple of the volume form of the Clifford algebra. Therefore, if we assume, as in Section 2.1 that this possibility is removed, then we again find that we require $\tilde{A} = 0$.

Generally, therefore, the constraints of higher order in the Clifford algebra, $(\tilde{\Phi}, \tilde{\Psi})$, tell us that the higher order variables, $(\tilde{A}, \tilde{\Pi})$, are redundant as far as the dynamics of the theory are concerned. Both the higher order constraints and the higher order variables may therefore be removed from the theory, and will not appear again in this paper. All of the spinorial variables and constraints left in the theory are purely first and second order in the Clifford algebra (from the $\mathbb{C}\mathrm{l}_{n-1}$ point of view). We thus are left with a theory with variables (A_i, e^i_I), conjugate momenta (Π^i, π^I_i), and constraints $(\Psi, \chi_{2i}, \chi_3, \Phi^i, \phi^I_i)$.

We should perhaps note that in dimension $2 \pmod 4$, if we did not remove by hand the unwanted terms from the theory, then the division into first and second class constraints given breaks down in the case of the highest order constraints. This leaves exactly the extra freedom in the definition of the connection given in Section 2.1 and the corresponding extra freedom in the conjugate momentum. What this extra freedom corresponds to in gravitational terms is not apparent, since the effect is spinorial in nature and would not occur in a purely tensorial approach.

3.3. **Minimal Theory.** Assuming linear independence of the vector fields $\mathbf{e}_I$ on Σ, we decompose the remaining variables (A_i, Π^i) and constraints (Ψ, Φ^i) using the densitised γ-matrices $\tilde{\sigma}^i$ by defining

$$A_i{}^j := \mathrm{Tr}\left[A_i \tilde{\sigma}^j\right], \qquad A_i{}^{jk} := \mathrm{Tr}\left[A_i \tilde{\sigma}^{[j}\tilde{\sigma}^{k]}\right],$$
$$\pi^{ij} := \mathrm{Tr}\left[\pi^i \tilde{\sigma}^j\right], \qquad \pi^{ijk} := \mathrm{Tr}\left[\pi^i \tilde{\sigma}^{[j}\tilde{\sigma}^{k]}\right],$$

and

$$\phi^{ij} := \mathrm{Tr}\left[\Phi^i \tilde{\sigma}^j\right], \qquad \phi^{ijk} := \mathrm{Tr}\left[\Phi^i \tilde{\sigma}^{[j}\tilde{\sigma}^{k]}\right],$$
$$\psi^i := \mathrm{Tr}\left[\Psi \tilde{\sigma}^i\right], \qquad \psi^{ij} := \mathrm{Tr}\left[\Psi \tilde{\sigma}^{[i}\tilde{\sigma}^{j]}\right].$$

The arguments of the previous section imply that, when we have eliminated the higher order quantities from the theory, we arrive at a theory with variables $(A_i{}^j, A_i{}^{jk}; \pi^{ij}, \pi^{ijk})$, constraints

$(\phi^{ij}, \phi^{ijk}, \psi^i, \psi^{ij}, \chi_{2i}, \chi_3)$ and Hamiltonian

$$H_T = -\int_\Sigma A_{ti}\psi^i + A_{tij}\psi^{ij} + N^i\chi_{2i} + \underline{N}\chi_3 + \int_\Sigma \lambda_{ijk}\phi^{ijk} + \lambda_{ij}\phi^{ij} + \lambda^i_I\phi^I_i. \tag{3.10}$$

We now wish to consider the constraints (ϕ^{ij}, ϕ^I_i). It is convenient to define linear combinations of the first set of constraints:

$$\phi^i_I = q^{-1/2} e_{Ij}\phi^{ij}.$$

It follows from the Poisson Bracket relations that the families of constraints $\{\phi^i_I\}$ and $\{\phi^I_i\}$ Poisson commute amongst themselves:

$$\left[\phi^i_I(x), \phi^j_J(y)\right] = 0, \qquad \left[\phi^I_i(x), \phi^J_j(y)\right] = 0, \tag{3.11}$$

but that there are non-trivial Poisson Brackets between the two families

$$\left[\phi^i_I(x), \phi^J_j(y)\right] = D\, q^{1/2}\left(\delta^J_I\delta^i_j - e^J_j e^i_I\right)\delta(x,y).$$

If $n \neq 2$, we can invert the operator on the right hand side of this equation to deduce that

$$\left(\delta^J_K\delta^k_j - (n-2)^{-1}e^k_K e^J_j\right)\left[\phi^i_I(x), \phi^K_k(y)\right] = D\, q^{1/2}\,\delta^J_I\,\delta^i_j\,\delta(x,y). \tag{3.12}$$

Therefore we deduce that, if $n \neq 2$, the matrix of Poisson Brackets of these two families of constraints is of maximal rank. Assuming linear independence of the vector fields $\mathbf{e}_I$, we may therefore consider the constraints ϕ^{ij} and ϕ^I_i as a conjugate set of second-class constraints to be removed from the theory. Constructing the inverse of the matrix of Poisson Brackets, and following the prescription of equation (3.6), we define the Dirac Bracket

$$\begin{aligned}[f(y), g(z)]^* = [f(y), g(z)] - \frac{1}{D}\int_\Sigma d^{n-1}x\, q^{-1/2}\left(\delta^J_I\delta^i_j - (n-2)^{-1}e^i_I e^J_j\right)& \\ \times\left\{\left[f(y), \phi^I_i(x)\right]\left[\phi^j_J(x), g(z)\right] - \left[f(y), \phi^j_J(x)\right]\left[\phi^I_i(x), g(z)\right]\right\}&.\end{aligned}$$

for any functions f and g on the phase space. With the Poisson Brackets redefined thus, the constraints (ϕ^I_i, ϕ^{ij}) have vanishing Poisson Brackets with all functions on the phase space. We are therefore free to impose the constraints ϕ^I_i, ϕ^{ij} as identities on the theory:

$$\pi^I_i = 0, \qquad \pi^{ij} = -D\, q\, q^{ij}.$$

These identities may be looked on as completely removing the variables π^I_i and π^{ij} from the theory. When we define the Poisson Brackets of the reduced theory with these variables removed as the Dirac Bracket above, we find that the only non-trivial Poisson Brackets of the remaining variables are

$$\left[A_i{}^j(x), A_k{}^l(y)\right] = \left(\delta^l_i A_k{}^j - A_i{}^l\delta^j_k\right)\delta(x,y), \tag{3.13a}$$

$$\left[A_i{}^j(x), \tilde\sigma^k(y)\right] = \delta^k_i\tilde\sigma^j\,\delta(x,y), \tag{3.13b}$$

$$\left[A_i{}^j(x), \pi^{klm}(y)\right] = \left(\delta^l_i\pi^{kjm} - \delta^m_i\pi^{kjl}\right)\delta(x,y), \tag{3.13c}$$

$$\left[A_i{}^j(x), A_k{}^{lm}(y)\right] = \left(\delta^l_i A_k{}^{jm} - \delta^m_i A_k{}^{jl}\right)\delta(x,y), \tag{3.13d}$$

$$\left[A_i{}^{jk}(x), \pi^{lmn}(y)\right] = -D\, q^2\,\delta^l_i\left(q^{jm}q^{kn} - q^{jn}q^{km}\right)\delta(x,y), \tag{3.13e}$$

where we have dropped the asterisks from the Dirac Bracket. These Poisson Brackets are the net effect of removing the second-class constraints (ϕ^I_i, ϕ^{ij}). Equations (3.13a) and (3.13b) imply that

$(A_i{}^j, \tilde{\sigma}^i)$ are not quite canonically conjugate variables, but obey relations analogous to variables $(q_i p^j, p^i)$ in ordinary mechanics, as is to expected from the definition of $A_i{}^j$.

Therefore, we have arrived at a "minimal" version of the general Hamiltonian theory with canonical variables

$$(A_i{}^j, \tilde{\sigma}^i, A_i{}^{jk}, \pi^{ijk}), \tag{3.14}$$

which obey the Poisson Bracket relations above. We have constraints

$$(\phi^{ijk}, \psi^i, \psi^{ij}, \chi_{2i}, \chi_3) \tag{3.15}$$

and time evolution is generated by the Hamiltonian

$$H_T = -\int_\Sigma A_{ti}\psi^i + A_{tij}\psi^{ij} + N^i\chi_{2i} + \underline{N}\chi_3 + \int_\Sigma \lambda_{ijk}\phi^{ijk}. \tag{3.16}$$

The behaviour of this theory in dimension 3, or in dimension 4 where we restrict to the positive chirality spin-bundle, is quite different from that in higher dimensions, or in dimension 4 where we work on the full spin-bundle. In the next sections, we will therefore first treat the two special cases, and then in Section 5 we consider the most general scenario.

4. Ashtekar Variables in Dimensions 3 and 4

Here we summarise the main simplifications that occur in dimensions 3 and 4. Since this material is, by now, standard [JS, S1, A2] we will be brief.

4.1. **Dimension 3.** If the manifold X is of dimension 3, we may take the γ-matrices to be proportional to the Pauli matrices

$$\gamma^0 = -i\sigma^3, \qquad \gamma^1 = -i\sigma^1, \qquad \gamma^2 = -i\sigma^2.$$

This means that

$$\gamma^{AB} = \epsilon^{ABC}\gamma_C,$$

where $\epsilon^{ABC} = \epsilon^{[ABC]}$ with $\epsilon^{012} = 1$ and internal indices are raised and lowered using the internal metric $(\eta_{AB}) = (\eta^{AB}) = \delta_{AB}$. Defining the two dimensional epsilon tensor $\epsilon^{IJ} = \epsilon^{0IJ}$ then

$$\Gamma^I = \gamma^{0I} = \epsilon^{IJ}\gamma_J.$$

In this case there is no distinction between pure elements of order 1 and 2 in the Clifford algebra. Therefore constraints of order 2 drop out of the theory, with the only constraints being the second-class constraints (ϕ^{ij}, ϕ_i^I) along with the constraints $(\chi_1, \chi_{2i}, \chi_3)$ which are first class. Removing the second-class constraints as above, we are left with a theory with canonically conjugate variables $(A_i, \tilde{\sigma}^j)$ and Hamiltonian

$$H_T = -\int_\Sigma \mathrm{Tr}\,(A_t\chi_1) + N^i\chi_{2i} + \underline{N}\chi_3, \tag{4.1}$$

The constraint χ_1 along with the equation of motion for the field $\tilde{\sigma}^i$ define the connection $\mathbf{A}$ to be the spinorial image of the spin-connection. Constraints χ_{2i} and χ_3 along with the equation of motion for the connection then tell us that this connection is flat. We therefore recover the usual description of Ricci-flat geometry in 3-dimensions. The quantisation of this theory has been discussed in some detail [A2, W2].

4.2. **Dimension 4 restricted to $\mathbb{V}^+$.** We define a set of 4-dimensional γ-matrices

$$\gamma^0 = \epsilon \otimes 1, \qquad \gamma^I = -i\sigma^1 \otimes \sigma^I,$$

where $\epsilon = i\sigma^2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We define the volume element

$$\omega = -\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^3 \otimes \mathrm{Id}.$$

Restricting to $\mathbb{V}^+ = \{\psi \in \mathbb{V} | \omega\psi = \psi\}$, then the restricted generators are

$$\Gamma^I = -i\sigma^I, \qquad \Gamma^{IJ} = \epsilon^{IJK}\Gamma_K,$$

where $\epsilon^{IJK} = \epsilon^{[IJK]}$ with $\epsilon^{123} = 1$. In this case, only the self-dual part of the spin-connection

$$A^+ = \frac{1}{2}\left(A^{0I} + \frac{1}{2}\epsilon^{IJK}A_{JK}\right)\Gamma_I$$

appears, and the constraints ϕ^{ijk} are again dropped from the theory. Removing the second-class constraints (ϕ^{ij}, ϕ^I_i) again leads to canonically conjugate variables $(A^+_i, \tilde{\sigma}^i)$, and the constraints of the theory are

$$\begin{aligned}
\chi_1 &= D^{A^+} \cdot \tilde{\sigma}, \\
\chi_{2i} &= -\mathrm{Tr}\left(F^{A^+}_{ij}\tilde{\sigma}^j\right), \\
\chi_3 &= \frac{1}{2}\mathrm{Tr}\left(F^{A^+}_{ij}\tilde{\sigma}^i\tilde{\sigma}^j\right).
\end{aligned}$$

This is the standard (Riemannian) version of the Ashtekar theory [A1, JS, S1].

5. General dimension

The theory we derived in Section 3.3 defined by variables (3.14), constraints (3.15) and Hamiltonian (3.16) has extra second class constraints in dimensions greater than 4, or in dimension 4 without restricting to the positive spin-bundle. We define the set of constraints

$$\rho^i := |q|^{-1} q_{jk}\phi^{jki}.$$

(In the notation of Section 3.2 this would correspond to $\Lambda^1[\mathrm{Tr}\,\phi]$ multiplied by $|q|^{-1}$). It follows from the Poisson Bracket relations (3.13) that constraints $\{\psi^i\}$ and $\{\rho^i\}$ Poisson commute amongst themselves

$$\left[\psi^i(x), \psi^j(y)\right] = 0, \qquad \left[\rho^i(x), \rho^j(y)\right] = 0,$$

but that there are non-trivial Poisson Brackets

$$\left[\psi^i(x), \rho^j(y)\right] = -2D\,(n-2)\,q\,q^{ij}\,\delta(x,y).$$

It therefore follows that, if $n \neq 2$, the two families of constraints $\{\psi^i\}$, $\{\rho^i\}$ are second-class, and their matrix of Poisson Brackets is of maximal rank. Following the Dirac procedure again, we deduce the new Dirac Bracket

$$\begin{aligned}
[f(y), g(z)]^* = [f(y), g(z)] - \frac{1}{2(n-2)D}\int_\Sigma d^{n-1}x\, q^{-1}\, q_{ij} \\
\times \left\{\left[f(y), \psi^i(x)\right]\left[\rho^j(x), g(z)\right] - \left[f(y), \rho^j(x)\right]\left[\psi^i(x), g(z)\right]\right\}.
\end{aligned}$$

for any functions f and g on the phase space.

In the notation of Section 3.2, the constraints (ψ^i, ρ^i) impose that the $\Lambda^1[\mathrm{Tr}\pi]$ momenta vanish, and that the variables $\Lambda^1[\mathrm{Tr}A]$ are defined by the equation

$$A_i{}^{ij} = -\frac{1}{2}\mathrm{Tr}\,(\tilde{\sigma}^j \partial_i \tilde{\sigma}^i). \tag{5.1}$$

Following Dirac's procedure, we therefore arrive at a theory with dynamical variables

$$(A_i{}^j, \tilde{\sigma}^i, \Lambda^3[\mathbf{A}], S^3_0[\mathbf{A}], \Lambda^3[\pi], S^3_0[\pi]),$$

with the variables $A_i{}^{ij}$ of the old theory now being derived quantities defined by equation (5.1). The Poisson Bracket relations (3.13) imply that the variables $(\tilde{\sigma}^i, \Lambda^3[\mathbf{A}], S^3_0[\mathbf{A}], \Lambda^3[\pi], S^3_0[\pi])$ Poisson commute with the constraints (ψ^i, ρ^i). Therefore, the Dirac Brackets involving these quantities are identically equal to the Poisson Brackets derived from the relations (3.13b)–(3.13e). The only remaining Dirac Bracket we need consider is therefore $\left[A_i{}^j(x), A_k{}^l(y)\right]^*$. Although the Poisson Brackets of the constraints with $A_i{}^j$ do not vanish, the Poisson Bracket of $A_i{}^j$ with the constraints $\{\rho^i\}$ vanishes weakly. Therefore, modulo constraints, this Dirac Bracket is equal to the Poisson Bracket (3.13a). It is easily checked that the remaining constraints of the theory, $(\psi^{ij}, \chi_{2i}, \chi_3, \Lambda^3[\phi], S^3_0[\phi])$, are all first class.

This is often the most useful form of the theory to work with if there are geometrical conditions one wishes to impose directly upon the curvature $\mathbf{F}$ or the connection $\mathbf{A}$. This is especially true if these geometrical conditions lead to the constraints χ_{2i} and χ_3 being satisfied automatically. A particular example of such a simplification is in the study of Ricci-flat Riemannian manifolds with reduced holonomy group. The reduction in holonomy group may be attributed to the existence of covariantly constant spinor fields [LM], which in turn leads to restrictions on the curvature of the spin-connection. These conditions may be integrated up, on a simply connected region, to restrictions on the spin-connection and, in this case, it is simplest to work with the full Hamiltonian theory (3.16) [G].

Alternatively, we may proceed to eliminate the remaining parts of the ϕ constraints from the theory. The constraints $(\Lambda^3[\phi], S^3_0[\phi])$ impose that the remaining momenta, $(\Lambda^3[\pi], S^3_0[\pi])$, vanish. Preservation of these conditions serves to define the remaining parts of the spatial connection $A_i{}^{jk}$ in terms of the vector fields $\mathbf{e}_I$, in a way which is consistent with this connection being identified with the (twice densitised) $(n-1)$-dimensional spin-connection.

Given that the preservation of the remaining constraints simply gives the definition of the remaining parts of the connection, it is possible to look on these definitions in themselves as extra constraints of the theory. These constraints would then be canonically conjugate to the remaining parts of the constraints ϕ^i, and therefore second-class. We can therefore remove all of these constraints from the theory, imposing them as identities. We are left with a theory with canonically conjugate variables $(A_i{}^j, \tilde{\sigma}^k)$ along with constraints $(\psi^{ij}, \chi_{2i}, \chi_3)$. We must, however, rewrite these constraints in terms of the canonical variables, replacing each part of the spatial connection $A_i{}^{jk}$ by its expression in terms of $\tilde{\sigma}^i$. This leaves us with a theory with Hamiltonian

$$H_T = -\int_\Sigma A_{tij}\psi^{ij} + N^i\chi_{2i} + \underline{N}\chi_3,$$

where we find that the constraints take the form

$$\begin{aligned}
\psi^{ij} &= \frac{1}{2}\left(A^{ij} - A^{ji}\right),\\
\chi_{2i} &= D_j A_i{}^j - D_i A_j{}^j,\\
\chi_3 &= \frac{D}{4}\left[\mathrm{Tr}\,\left(A^2\right) - \mathrm{Tr}\,(A)^2 + s(\mathbf{q})\right].
\end{aligned}$$

In this equations, we have used the soldering form to construct the twice densitised inverse metric with components

$$|\mathrm{det} q| q^{ij} = -\frac{\mathrm{Tr}\,(\tilde{\sigma}^i\tilde{\sigma}^j)}{D},$$

and $A^{ij} = q^{ik}A_k{}^j$. With this metric we then construct the Levi-Civita connection D and scalar curvature $s(\mathbf{q})$. The formalism has therefore reduced to a spinorial version of the standard ADM Hamiltonian theory. The field $A_i{}^j$ corresponds to a densitised version of the extrinsic curvature k_{ij}, and the soldering forms $\tilde{\sigma}^i$ are a densitised spinorial version of the vector fields $\mathbf{e}_I$.

The Poisson Bracket relations (3.13b) imply that all of the constraints are first-class. Counting degrees of freedom, we have $2(n-1)^2$ variables, $(A_i{}^j, \tilde{\sigma}^k)$, and $\frac{1}{2}(n-1)(n-2) + (n-1) + 1$ first-class constraints. Therefore, we have $n(n-3)$ Hamiltonian degrees of freedom, as expected.

6. Conclusion

We have given a spinorial set of Hamiltonian variables for General Relativity which work in any dimension greater than 2. Although, for simplicity, we have restricted ourselves to Riemannian signature, a similar analysis carries through in any signature, with appropriate minus signs.

In dimensions $0 \pmod 4$, the theory can be reduced to the positive chirality spin-bundle, but this is not possible in dimensions $2 \pmod 4$ for algebraic reasons. It is noticeable that the theory is very different in the special cases of 3 dimensions and in 4 dimensions when restricted to the positive spin-bundle. In these cases, the constraint χ_1 is first-class, and the ϕ^i constraints have no effect other than setting $\pi^i = \tilde{\sigma}^i$. In the more general case, the constraints ϕ^i are conjugate to parts of the χ_1 constraints, and we are forced to remove part of the spatial part of the connection from the problem. As in the Palatini formalism, it is this removal of the extra constraints which leads to the apparent non-polynomial nature of remaining constraints [A2]. Whether it is useful to reduce the theory to ADM form seems to depend on the type of problems we wish to tackle. If we wish to impose geometrical conditions directly on the connection or its curvature, it seems more useful to proceed without removing the extra constraints first. This may also be the more useful course if one wishes to quantise the theory.

One obvious problem would be to consider the extension of this approach to canonical supergravity theories. Certainly four-dimensional minimal supergravity can be written in Ashtekar-type form [J], and recent attempts to find a unified approach to the gravitational field and the 3-form potential of 11-dimensional supergravity seem to suggest links with Ashtekar variables [MN]. It also seems possible that our approach may be useful in dimensions $0 \pmod 4$ when one considers theories with chiral fermions.

A more geometrical problem mentioned earlier concerns the description of Ricci-flat Riemannian metrics with reduced holonomy group. The reductions of the holonomy group to Ricci-flat Kähler (dimension $n = 2k$), hyper-Kähler ($n = 4k$), and G_2 ($n = 7$) or Spin_7 ($n = 8$) may be described

in terms of the existence of covariantly constant spinors of various types on a manifold [LM]. An analysis of this problem in light of the current formalism will be given elsewhere [G].

Acknowledgements. This work was funded by the EPSRC.

References

[AW] Alvarez-Gaumé L. and Witten E.: Gravitational Anomalies. Nuclear Physics **B234**, 269–330 (1983).

[A1] Ashtekar A.: New variables for classical and quantum gravity. Physical Review Letters **57**, 2244–7 (1986).

[A2] Ashtekar A.: Lectures on Non-Perturbative Canonical Gravity. Singapore: World Scientific 1991.

[D] Dirac P.A.M.: Lectures on Quantum Mechanics. Belfer Graduate School of Science. New York: Yeshiva University 1964.

[G] Grant J.D.E.: A spinorial approach to Ricci-flat Riemannian manifolds with reduced holonomy group. In preparation.

[J] Jacobson T.: New variables for canonical supergravity. Classical and Quantum Gravity **5**, 923–35 (1988).

[JS] Jacobson T. and Smolin L.: The left-handed spin-connection as a variable for quantum gravity. Physics Letters **196B**, 39–42 (1987).

[LM] Lawson H.B., Michelsohn M.-L.: Spin Geometry. Oxford: Princeton University Press 1989.

[MN] Melosch S. and Nicolai H.: New canonical variables for $d = 11$ supergravity. Physics Letters **B416**, 91–100 (1998).

[PR] Penrose R., Rindler W.: Spinors and Spacetime, volume 2. Cambridge: Cambridge University Press 1986.

[S1] Samuel J.: A Lagrangian Basis for Ashtekar's Reformulation of Canonical Gravity. Pramana J. Phys. **28**, L429–31 (1987).

[S2] Sen A.: On the existence of neutrino zero modes in vacuum space-times. Journal of Mathematical Physics **22**, 1718–26 (1981).

[W1] Witten E.: A New Proof of the Positive Energy Theorem. Communications in Mathematical Physics **80** 381–402 (1981).

[W2] Witten E.: $2+1$ Dimensional Gravity as an Exactly Soluble System. Nuclear Physics **B311** 46–78 (1988/89).

Department of Mathematics, University of Hull, Hull HU6 7RX, U.K.